Spinor Operator Giving Both Angular Momentum and Parity

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In heavy quark effective theory, heavy mesons which contain a heavy quark (or antiquark) are classified by $s_\ell^{\pi_\ell}$, i.e., the total angular momentum s_ℓ and the parity π_ℓ of the light quark degrees of freedom around a static heavy quark. In this case, however, one needs to separately estimate the parity other than the angular momentum of a light quark to describe heavy mesons.

A new operator K was proposed some time ago by two of us (T.M. and T.M.). In this Letter, we show that the quantum number k of this operator is enough to describe both the total angular momentum of the light quark degrees of freedom and the parity of a heavy meson, and derive a simple relation between k and $s_\ell^{\pi_\ell}$.

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Recent discovery of narrow meson states $D_{sJ}(2317)$ and $D_{sJ}(2460)$ by BaBar and the following confirmation by CLEO and Belle [1] has triggered a series of study on spectroscopy of heavy mesons again. Though $D_{sJ}(2317)$ and $D_{sJ}(2460)$ are assigned as $j^P = 0^+$ and 1^+, respectively, their masses are significantly smaller than the predictions based on many of potential models [2]. To explain these masses, Bardeen, Eichten, Hill and others [3, 4] proposed an interesting idea of an effective Lagrangian with chiral symmetries of light quarks and heavy quark symmetry. The heavy meson states with the total angular momentum $j = 0$ and $j = 1$ related to s_ℓ (the total angular momentum of the light quark degrees of freedom) $= 1/2$ make the parity doublets $(0^-, 0^+)$ and $(1^-, 1^+)$, respectively, and the members in these doublets degenerate in the limit of chiral symmetry. Furthermore, the two states $(0^-, 1^-)$ degenerate in the limit of heavy quark symmetry, as well as $(0^+, 1^+)$. These doublets are called the heavy spin multiplets.

These newly discovered states are well classified in heavy quark effective theory, i.e., in terms of $s_\ell^{\pi_\ell}$, where s_ℓ and π_ℓ represent the total angular momentum and the parity of the light quark degrees of freedom around a static heavy quark, respectively. In this case, however, one has to separately estimate the parity and the angular momentum of a light quark for each heavy meson state.

Some time ago, two of the authors (T.M. and T.M.) proposed a new bound state equation for atomlike mesons, i.e., heavy mesons composed of a heavy quark and a light antiquark, and they also proposed a new operator K which can classify heavy mesons well [5]. In this Letter, we show that this operator K, given by Eq. (5) below, has the information about not only s_ℓ but also the parity of heavy mesons and naturally explains the heavy spin multiplets. That is, only the quantum number k corresponding to the operator K can reproduce both the total angular momentum of the light quark degrees of freedom *and* the parity of a heavy meson. We also discuss the relation between k and $s_\ell^{\pi_\ell}$.

Let us consider a heavy meson composed of a heavy quark Q and a light antiquark $\bar{q}$. The effective Hamiltonian of this system is obtained by applying the Foldy-Wouthuysen-Tani (FWT) transformation to the heavy quark Q. One can formulate the equation so that we can cast the structure of the eigenvalue equation into a simple form and make the Dirac-like equation in the large limit of the heavy quark mass m_Q [5]. In order to show why we can introduce a new operator K for heavy mesons, we consider the equation with $1/m_Q$ corrections neglected, whose contribution should be important in numerical analysis of spectroscopy.

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The lowest energy for the $Q\bar{q}$ bound state is given by $m_Q + E_0^a$ after solving the equation [5]

$$H_0 \otimes \psi_0^a = E_0^a \psi_0^a, \qquad H_0 = \vec{\alpha}_q \cdot \vec{p}_q + \beta_q \left(m_q + S(r)\right) + V(r), \tag{1}$$

where a expresses all the quantum numbers and quantities with the subscript q mean those for a light antiquark. $S(r)$ is a confining scalar potential and $V(r)$ is a Coulombic vector potential at short distances. Both potentials have dependence only on r, the relative distance between Q and $\bar{q}$. With a symbol $\otimes$, one should note that gamma matrices for a light antiquark be multiplied from left with the wave function while those for a heavy quark from right.

Using the 2×2 matrix eigenfunctions $y_{j\,m}^k$ of angular part defined below and the radial functions f_k and g_k, the 4×4 matrix solution to Eq. (1) is given by [5]

$$\psi_0^a = \left(0 \quad \Psi_{j\,m}^k(\vec{r})\right), \tag{2}$$

$$\Psi_{j\,m}^k(\vec{r}) = \frac{1}{r}\begin{pmatrix} f_k(r)\, y_{j\,m}^k \\ i g_k(r)\, y_{j\,m}^{-k} \end{pmatrix}, \tag{3}$$

where j and m are the total angular momentum of a heavy meson and its z-component, respectively. The total angular momentum of a heavy meson is the sum of the total angular momentum of the light quark degrees of freedom $\vec{S}_\ell$ and the heavy quark spin $\frac{1}{2}\vec{\Sigma}_Q$:

$$\vec{J} = \vec{S}_\ell + \frac{1}{2}\vec{\Sigma}_Q \qquad \text{with} \quad \vec{S}_\ell = \vec{L} + \frac{1}{2}\vec{\Sigma}_q, \tag{4}$$

where $\frac{1}{2}\vec{\Sigma}_q$ $(= \frac{1}{2}\vec{\sigma}_q\ 1_{2\times 2})$ and $\vec{L}$ are the 4-component spin and the orbital angular momentum of a light antiquark, respectively. Furthermore, k is the quantum number of the spinor operator K, which was introduced in Eq. (20) of Ref. [5], defined by

$$K = -\beta_q\left(\vec{\Sigma}_q \cdot \vec{L} + 1\right), \qquad K\,\Psi_{j\,m}^k = k\,\Psi_{j\,m}^k. \tag{5}$$

It is interesting to note that the same form of the operator K is defined in the case of a single Dirac particle in a central potential [6]. It is remarkable that in our approach K can be defined even for a heavy meson which is a two-body bound system composed of a heavy quark and a light antiquark.

Here we show that there is a relation between k and s_ℓ, being often used in heavy quark effective theory. Let us calculate the square of K.

$$\begin{aligned} K^2 &= \left(\Sigma_q\right)_i \left(\Sigma_q\right)_j L_i L_j + 2\vec{\Sigma}_q \cdot \vec{L} + 1 = \vec{L}^2 + \vec{\Sigma}_q \cdot \vec{L} + 1 \\ &= \vec{S}_\ell^2 + \frac{1}{4}. \end{aligned} \tag{6}$$

Therefore, the operator K^2 is equivalent to $\vec{S}_\ell^2$ and it holds

$$k = \pm\left(s_\ell + \frac{1}{2}\right) \quad \text{or} \quad s_\ell = |k| - \frac{1}{2}. \tag{7}$$

Now, let us briefly summarize the properties of the eigenfunctions $y_{j\,m}^k$, whose details are given in [5]. To begin with, we need to introduce the so-called vector spherical harmonics which are defined by [7]

$$\vec{Y}_{j\,m}^{(\mathrm{L})} = -\vec{n}\,Y_j^m, \quad \vec{Y}_{j\,m}^{(\mathrm{E})} = \frac{r}{\sqrt{j(j+1)}}\vec{\nabla} Y_j^m, \quad \vec{Y}_{j\,m}^{(\mathrm{M})} = -i\vec{n}\times\vec{Y}_{j\,m}^{(\mathrm{E})}, \tag{8}$$

where Y_j^m are the spherical polynomials and $\vec{n} = \vec{r}/r$. These vector spherical harmonics are nothing but a set of eigenfunctions for a spin-1 particle. $\vec{Y}_{j\,m}^{(\mathrm{A})}$ (A=L, M, E) are eigenfunctions of $\vec{J}^2$ and J_z, having the eigenvalues $j(j+1)$ and m. The parities are assigned as $(-)^{j+1}$, $(-)^j$, $(-)^{j+1}$ for A=L, M, E, respectively, since Y_j^m has a parity $(-)^j$.

In order to diagonalize the leading Hamiltonian of Eq. (1) in the k space, it is necessary to make $\vec{Y}_{j\,m}^{(\mathrm{A})}$ and Y_j^m into the spinor representation $y_{j\,m}^k$ by the following unitary transformation

$$\begin{pmatrix} y_{j\,m}^{-(j+1)} \\ y_{j\,m}^{j} \end{pmatrix} = U\begin{pmatrix} Y_j^m \\ \vec{\sigma}\cdot\vec{Y}_{j\,m}^{(\mathrm{M})} \end{pmatrix}, \qquad \begin{pmatrix} y_{j\,m}^{j+1} \\ y_{j\,m}^{-j} \end{pmatrix} = U\begin{pmatrix} \vec{\sigma}\cdot\vec{Y}_{j\,m}^{(\mathrm{L})} \\ \vec{\sigma}\cdot\vec{Y}_{j\,m}^{(\mathrm{E})} \end{pmatrix}, \tag{9}$$

TABLE I: States classified by various quantum numbers

j^P	0^-	1^-	0^+	1^+	1^+	2^+	1^-
k	-1	-1	1	1	-2	-2	2
$s_\ell^{\pi_\ell}$	$\frac{1}{2}^-$	$\frac{1}{2}^-$	$\frac{1}{2}^+$	$\frac{1}{2}^+$	$\frac{3}{2}^+$	$\frac{3}{2}^+$	$\frac{3}{2}^-$
${}^{2s+1}l_j$	${}^1S_0$	${}^3S_1$	${}^3P_0$	${}^3P_1$, ${}^1P_1$	${}^1P_1$, ${}^3P_1$	${}^3P_2$	${}^3D_1$
Ψ_j^k	Ψ_0^{-1}	Ψ_1^{-1}	Ψ_0^1	Ψ_1^1	Ψ_1^{-2}	Ψ_2^{-2}	Ψ_1^2

where the orthogonal matrix U is introduced as

$$U = \frac{1}{\sqrt{2j+1}} \begin{pmatrix} \sqrt{j+1} & \sqrt{j} \\ -\sqrt{j} & \sqrt{j+1} \end{pmatrix}. \tag{10}$$

$y_{j\,m}^k$ are 2×2 matrix eigenfunctions of three operators, $\vec{J}^2$, J_z, and $\vec{\sigma}_q \cdot \vec{L}$ with eigenvalues, $j(j+1)$, m, and $-(k+1)$, respectively, and have the interesting properties

$$(\vec{\sigma}_q \cdot \vec{n}) \otimes y_{j\,m}^k \;=\; -y_{j\,m}^{-k}, \tag{11}$$

$$(\vec{\sigma}_q \cdot \vec{L}) \otimes y_{j\,m}^k \;=\; -(k+1)\; y_{j\,m}^k, \tag{12}$$

where the quantum number k can take only values as shown in Eq. (7)

$$k = \pm j \quad \text{or} \quad \pm (j+1). \tag{13}$$

It should be noted that k is nonzero since $\vec{Y}_{0\,0}^{(\mathrm{M})}$ does not exist.

Substituting Eqs. (2) and (3) into Eq. (1) and using Eqs. (11) and (12), one can eliminate the angular part of the wave function, $y_{j\,m}^k$, and obtain the radial equation as follows,

$$\begin{pmatrix} m_q + S + V & -\partial_r + \frac{k}{r} \\ \partial_r + \frac{k}{r} & -m_q - S + V \end{pmatrix} \Psi_k(r) = E_0^k\; \Psi_k(r), \qquad \Psi_k(r) \equiv \begin{pmatrix} f_k\,(r) \\ g_k\,(r) \end{pmatrix}, \tag{14}$$

which depends on k alone. This is quite similar to a one-body Dirac equation in a central potential. Since K commutes with H_0 and the states $\Psi_{j\,m}^k$ have the same energy E_0^k with different values of j, these states are degenerate with the same value of k at the lowest order in $1/m_Q$.

The parity P' of the eigenfunction ψ_0^a is determined by the upper ("large") two-by-two components $y_{j\,m}^k$ as

$$P' = \begin{cases} (-1)^j & \text{for} \quad \Psi_{j\,m}^{-(j+1)} \;\text{ and }\; \Psi_{j\,m}^{j} \\ (-1)^{j+1} & \text{for} \quad \Psi_{j\,m}^{j+1} \;\text{ and }\; \Psi_{j\,m}^{-j} \end{cases} \tag{15}$$

Thus, using the relations of Eqs. (13) and (15) and taking into account the intrinsic parity of the light antiquark, one can simply write the parity of a heavy meson as

$$P = -P' = \frac{k}{|k|}(-1)^{|k|+1} \tag{16}$$

Notice that the parity P of the whole system is equal to the parity π_ℓ of the light quark degrees of freedom, as can be seen in TABLE I, since the intrinsic parity of a heavy quark is $+1$.

In heavy quark effective theory, heavy mesons are normally classified in terms of $s_\ell^{\pi_\ell}$, since at the lowest order heavy quarks in those mesons are considered to be static, namely it stays rest at the center of a heavy meson system. In this work, we have found that (i) the parity of a heavy meson and (ii) the total angular momentum of the light quark degrees of freedom can be reproduced in terms of k alone as seen from Eqs. (16) and (7), respectively. We have also found that the degeneracy between members in each heavy spin multiplet, $(0^-,\ 1^-)$ and $(0^+,\ 1^+)$, is automatic in our approach [5], while the method using the effective Lagrangian with heavy-quark as well as chiral symmetries must force degeneracy among parity doublets to construct such a Lagrangian [3, 4]. These are *the main results of this paper.*

As our summary, several states are classified by various quantum numbers in TABLE I. The states with different j but with the same parity P make a heavy spin multiplet of heavy mesons, which corresponds to *heavy quark symmetry* in heavy quark effective theory. One can see that k naturally explains the heavy spin doublets.

Before closing our discussions, we comment about k from the phenomenological point of view. The lowest order solution satisfies degeneracy in k since the energy depends only on k, i.e., $j^P = 0^-$ and 1^- states have the same mass, so are the 0^+ and 1^+ states. This degeneracy is resolved by including higher order terms in $1/m_Q$ [5] and one can phenomenologically discuss mass spectra of these heavy mesons even though some objections [8] for using a potential model exist. A comprehensive analysis on mass spectra of heavy mesons including $D_{sJ}(2317)$ and $D_{sJ}(2460)$ is in progress.

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